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Oscillation Theorems for Second Order Nonlinear Differential Equations

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Some oscillation criteria are established for certain second order nonlinear differential equations of the form

$$(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\,f(x(t)) = 0.$$

These criteria improve upon some of the known results by Kura, Kamenev and Philos.

1. INTRODUCTION

Consider the nonlinear second order differential equations

$$(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\,f(x(t)) = 0, \quad \left(\cdot = \frac{d}{dt}\right), \tag{1}$$

and

$$(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\,f(x(t)) = b(t), \tag{2}$$

where a, b, p, q: $[t_0, \infty) \to R = (-\infty, \infty)$, ψ, f: $R \to R$ are continuous, $a(t) > 0$ and $xf(x) > 0$ for $x \neq 0$.

We restrict our attention to those solutions of Eqs. (1) and (2) which exist on some ray $[t_1, \infty)$, where $t_1 \geqslant t_0$ and which are nontrivial in any neighborhood of infinity. Such a solution is called oscillatory if it has

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arbitrarily large zeros. Otherwise the solution is called nonoscillatory. An equation is called oscillatory if all its solutions are oscillatory.

Many authors considered the second order sublinear differential equation

$$x''(t)+q(t)|x(t)|^{\alpha}\operatorname{sgn} x(t)=0, \qquad 0<\alpha<1, \tag{3}$$

where $q\colon [t_0, \infty)\to R$ is continuous, and established sufficient conditions so that all solutions of Eq. (3) are oscillatory. Some of these conditions are:

$$q(t)\geqslant 0, \int_{t_0}^{\infty} s^{\alpha}q(s)\,ds=\infty \qquad \text{(Belohorec [1])}, \tag{4}$$

$$\int_{t_0}^{\infty} s^{\beta}q(s)\,ds=\infty \qquad \text{for some } \beta\in[0,\alpha] \text{ (Belohorec [2])}, \tag{5}$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)\,d\tau\,ds=\infty \qquad \text{(Kamenev [6])}, \tag{6}$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} \tau^{\beta}q(\tau)\,d\tau\,ds=\infty \quad \text{for some } \beta\in[0,\alpha], \text{ (Kura [7])}. \tag{7}$$

Kura's criterion (7) improves those of Belohorec and Kamenev. Philos [8] extends Kura's result to more general equations of the form

$$x''(t)+p(t)\,x'(t)+q(t)\,f(x(t))=0, \tag{8}$$

where p, $q\colon [t_0, \infty)\to R$, $f\colon R\to R$ are continuous, $xf(x)>0$, $f'(x)>0$ for $x\neq 0$ and f is strongly sublinear, i.e., $\int_{\pm 0} du/f(u)<\infty$. The oscillatory behavior of the equation

$$(a(t)\,x'(t))'+q(t)\,f(x(t))=0, \tag{9}$$

where $a\colon [t_0, \infty)\to(0,\infty)$, $q\colon [t_0,\infty)\to R$, and $f\colon R\to R$ are continuous, $xf(x)>0$ and $f'(x)\geqslant 0$ for $x\neq 0$ and $\int^{\infty}(1/a(s))\,ds=\infty$, was discussed by Bhatia [3]. He proved that the condition

$$\int^{\infty} q(s)\,ds=\infty \tag{10}$$

is sufficient for all solutions of Eq. (9) to be oscillatory. Bhatia's result was extended by Graef, Rankin, and Spikes [5] and the present authors [4] to more general equations of the form (1) with $\psi(x)=1$.

The purpose of this paper is to extend and considerably improve the above-mentioned results of Belohorec, Bhatia, Kamenev, Kura, and Philos. We also impose conditions under which every solution $x(t)$ of Eq. (2) has $\lim_{t\to\infty} \inf x(t) = 0$. The result obtained generalizes and unifies the one obtained by Philos.

2. Main Results

The following theorem is concerned with the oscillatory behavior of Eq. (1), where the damping coefficient $p(t)$ is nonpositive.

THEOREM 1. *Let $p(t) \leqslant 0$ for $t \geqslant t_0$, $\psi(x) \geqslant c > 0$, $f'(x)/\psi(x) \geqslant \alpha > 0$, for $x \neq 0$, and*

$$\int^{\infty} \frac{1}{a(s)}\, ds = \infty. \tag{11}$$

Suppose that there is a differentiable function ρ: $[t_0, \infty) \to (0, \infty)$ such that $\dot{\rho}(t) \geqslant 0$ and

$$\int^{\infty} \rho(s) \left[q(s) - \frac{a(s)}{4\alpha} \left(\frac{\dot{\rho}(s)}{\rho(x)} - \frac{p(s)}{ca(s)} \right)^2 \right] ds = \infty, \tag{12}$$

then Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1), say $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Put

$$\omega(t) = \rho(t) \frac{a(t)\, \psi(x(t))\, \dot{x}(t)}{f(x(t))}.$$

Then

$$\begin{aligned}
\dot{\omega}(t) &= \rho(t) \frac{((a(t)\, \psi(x(t))\, \dot{x}(t))^{\cdot}}{f(x(t))} + \dot{\rho}(t) \frac{a(t)\, \psi(x(t))\, \dot{x}(t)}{f(x(t))} \\
&\quad - \frac{\rho(t)\, a(t)\, \psi(x(t))\, f'(x(t))\, \dot{x}^2(t)}{f^2(x(t))} \\
&= -\rho(t)\, q(t) + \left(\frac{\dot{\rho}(t)}{\rho(t)} - \frac{p(t)}{a(t)\, \psi(x(t))} \right) \omega(t) \\
&\quad - \frac{f'(x(t))}{a(t)\, \rho(t)\, \psi(x(t))}\, \omega^2(t)
\end{aligned}$$

$$= -\rho(t)\,q(t) + \frac{a(t)\,\rho(t)\,\psi(x(t))}{4f'(x(t))}\left[\frac{\rho^{\cdot}(t)}{\rho(t)} - \frac{p(t)}{a(t)\,\psi(x(t))}\right]^2$$

$$-\left[\left(\frac{f'(x(t))}{a(t)\,\rho(t)\,\psi(x(t))}\right)^{1/2}\omega(t) - \frac{\left[\dfrac{\rho^{\cdot}(t)}{\rho(t)} - \dfrac{p(t)}{a(t)\,\psi(x(t))}\right]}{2\left(\dfrac{f'(x(t))}{a(t)\,\rho(t)\,\psi(x(t))}\right)^{1/2}}\right]^2$$

$$\leqslant -\rho(t)\,q(t) + \frac{a(t)\,\rho(t)}{4\alpha}\left[\frac{\rho^{\cdot}(t)}{\rho(t)} - \frac{p(t)}{ca(t)}\right]^2.$$

Integrating the above inequality from t_1 to t $(\geqslant t_1)$, we obtain

$$\rho(t)\,\frac{a(t)\,\psi(x(t))\,x^{\cdot}(t)}{f(x(t))}$$

$$\leqslant \rho(t_1)\,\frac{a(t_1)\,\psi(x(t_1))\,x^{\cdot}(t_1)}{f(x(t_1))}$$

$$-\int_{t_1}^{t} \rho(s)\left[q(s) - \frac{a(s)}{4\alpha}\left[\frac{\rho^{\cdot}(s)}{\rho(s)} - \frac{p(s)}{ca(s)}\right]^2\right] ds. \tag{13}$$

In view of condition (12), it follows from (13) that there exists a $t_2 \geqslant t_1$ such that $x^{\cdot}(t) < 0$ for $t \geqslant t_2$. It also follows from (12) that there exists a $t_3 \geqslant t_2$ such that

$$\int_{t_2}^{t_3} \rho(s)\,\gamma(s)\,ds = 0 \quad \text{and} \quad \int_{t_3}^{t} \rho(s)\,\gamma(s)\,ds \geqslant 0 \quad \text{for} \quad t \geqslant t_3,$$

where

$$\gamma(t) = q(t) - \frac{a(t)}{4\alpha}\left[\frac{\rho^{\cdot}(t)}{\rho(t)} - \frac{p(t)}{ca(t)}\right]^2.$$

Integrating Eq. (1) by parts we have

$$a(t)\,\psi(x(t))\,x^{\cdot}(t) = a(t_3)\,\psi(x(t_3))\,x^{\cdot}(t_3) - \int_{t_3}^{t} p(x)\,x^{\cdot}(s)\,ds - \int_{t_3}^{t} q(s)\,f(x(s))\,ds.$$

Since $p(t) \leqslant 0$, we obtain

$$a(t)\,\psi(x(t))\,x^{\cdot}(t)$$

$$\leqslant a(t_3)\,\psi(x(t_3))\,x^{\cdot}(t_3)$$

$$-\int_{t_3}^{t} \rho(s)\left[q(s) - \frac{a(s)}{4\alpha}\left[\frac{\rho^{\cdot}(s)}{\rho(s)} - \frac{p(s)}{ca(s)}\right]^2\right]\left(\frac{f(x(s))}{\rho(s)}\right) ds$$

$$-\int_{t_3}^{t} \frac{a(s)}{4\alpha}\left[\frac{\rho^{\cdot}(s)}{\rho(s)} - \frac{p(s)}{ca(s)}\right]^2 f(x(s))\,ds$$

$$
\begin{aligned}
&\leqslant a(t_3)\,\psi(x(t_3))\,x^{\cdot}(t_3)-\frac{f(x(t))}{\rho(t)}\\
&\quad\times\int_{t_3}^{t}\rho(s)\left[q(s)-\frac{a(s)}{4\alpha}\left[\frac{\rho^{\cdot}(s)}{\rho(s)}-\frac{p(s)}{ca(s)}\right]^2\right]ds\\
&\quad+\int_{t_3}^{t}\left[\frac{\rho(s)\,f'(x(s))\,x^{\cdot}(s)-\rho^{\cdot}(s)\,f(x(s))}{\rho^2(s)}\right]\\
&\quad\times\int_{t_3}^{s}\rho(\tau)\left[q(\tau)-\frac{a(\tau)}{4\alpha}\left[\frac{\rho^{\cdot}(\tau)}{\rho(\tau)}-\frac{p(\tau)}{ca(\tau)}\right]^2\right]d\tau\,ds\\
&\leqslant a(t_3)\,\psi(x(t_3))\,x^{\cdot}(t_3).
\end{aligned}
$$

Thus

$$\int_{t_3}^{t}\psi(x(s))\,x^{\cdot}(s)\,ds\leqslant a(t_3)\,\psi(x(t_3))\,x^{\cdot}(t_3)\int_{t_3}^{t}\frac{1}{a(s)}\,ds.$$

By virtue of (11) and the fact that $x^{\cdot}(t_3)<0$, it follows that $\int_{t_3}^{t}\psi(x(s))\,x^{\cdot}(s)\,ds\to-\infty$ as $t\to\infty$, which contradicts the assumption that $x(t)>0$ for $t\geqslant t_1$. The proof of the case that $x(t)<0$ for $t\geqslant t_1$ is similar and hence is omitted.

Remark. If $\psi(x)=1$, then the damping coefficient $p(t)$ need not be of fixed sign, and when $\rho(t)=1$, the condition $f'(x)/\psi(x)\geqslant\alpha>0$ can be replaced by $f'(x)\geqslant 0$ for $x\neq 0$. Thus our Theorem 1 includes Theorem 1 in [3], Theorem 1 in [4] and Theorem 1 in [5]. For illustration we consider the equation

$$x^{\cdot\cdot}(t)+\frac{1}{t^{\alpha_1}}\left(\frac{1}{t}+\sin t\right)x(t)=0,\qquad t>0. \tag{14}$$

Each one of Theorem 1 in [3], Theorem 1 in [4] and Theorem 1 in [5] ensures the oscillation of Eq. (14) only if $\alpha_1=0$ while Theorem 1 implies the oscillation of Eq. (14) for $\alpha_1=1$.

Here are some further illustrative examples.

EXAMPLE 1. Consider the equation

$$(t^{1/2}(2-\sin x)\,x^{\cdot})^{\cdot}-t^{-1/2}x^{\cdot}+\frac{1}{t}\frac{\sin t}{2-\sin t}\,x=0,\qquad t>0. \tag{15}$$

Here

$$\frac{f'(x)}{\psi(x)}=\frac{1}{2-\sin x}\geqslant\frac{1}{3}=\alpha>0.$$

All conditions of Theorem 1 are satisfied where we take $\rho(t)=t$, and hence Eq. (15) is oscillatory.

EXAMPLE 2. Consider the equation

$$(\sqrt{t}\,\ln(e+x^2)\,x^{\cdot})^{\cdot} - t^{-1/2}x^{\cdot} + \frac{1}{t}\left(\frac{1}{t}+\sin t\right)(x+x^3)=0, \qquad t>0. \quad (16)$$

Here

$$\frac{f'(x)}{\psi(x)} = \frac{1+3x^2}{\ln(e+x^2)} \geqslant 1 = \alpha > 0.$$

All conditions of Theorem 1 are satisfied for $\rho(t)=t$, and hence all solutions of Eq. (16) are oscillatory.

We believe that the oscillatory behavior of Eqs. (15) and (16) is not deducible from known oscillation criteria.

THEOREM 2. *Let $\psi(x)\geqslant c>0$, $f'(x)>0$ for $x\neq 0$,*

$$\int_{+0} \frac{\psi(u)}{f(u)}\,du < \infty \qquad \text{and} \qquad \int_{-0} \frac{\psi(u)}{f(u)}\,du < \infty, \quad (17)$$

and assume that there is a twice differentiable function ρ: $[t_0, \infty) \to (0, \infty)$ such that

$$\rho^{\cdot\cdot}(t)<0, \quad p(t)\left(\frac{\rho^{\cdot}(t)}{\rho(t)} + \frac{a(t)}{\rho(t)}\left(\frac{\rho(t)}{a(t)}\right)^{\cdot}\right) \leqslant 0 \quad \textit{for } t \geqslant t_0 \quad (18)$$

and

$$\sup_{t\geqslant t_0}\left(-\frac{\rho(t)}{\rho^{\cdot\cdot}(t)}\gamma(t)\right) \leqslant \min\left\{\inf_{x>0}\frac{F(x)\,f'(x)}{\psi(x)},\ \inf_{x<0}\frac{F(x)\,f'(x)}{\psi(x)}\right\}, \quad (19)$$

where

$$\gamma(t) = \left[\frac{\rho^{\cdot}(t)}{\rho(t)} + \frac{a(t)}{\rho(t)}\left(\frac{\rho(t)}{a(t)}\right)^{\cdot} - \frac{1}{c}\frac{p(t)}{a(t)}\right]^2 \quad \textit{and} \quad F(x) = \int_0^x \frac{\psi(u)\,du}{f(u)}.$$

If

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\frac{\rho(\tau)}{a(\tau)}\,q(\tau)\,d\tau\,ds = \infty, \quad (20)$$

then Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Assume $x(t)>0$ for $t\geqslant t_1\geqslant t_0$. Put

$$\omega(t)=\rho(t)\,F(x(t)).$$

It is easy to verify that

$$\begin{aligned}\omega^{\cdot\cdot}(t)&=\rho^{\cdot\cdot}(t)\,F(x(t))+\left[\rho^{\cdot}(t)+a(t)\left(\frac{\rho(t)}{a(t)}\right)^{\cdot}\right]\frac{\psi(x(t))\,x^{\cdot}(t)}{f(x(t))}\\&\quad+\frac{\rho(t)}{a(t)}\frac{(a(t)\,\psi(x(t))\,x^{\cdot}(t))^{\cdot}}{f(x(t))}-\rho(t)\frac{\psi(x(t))}{f^2(x(t))}f'(x(t))\,x^{\cdot 2}(t)\\&=-\frac{\rho(t)}{a(t)}q(t)+\left[\frac{\rho^{\cdot}(t)}{\rho(t)}+\frac{a(t)}{\rho(t)}\left(\frac{\rho(t)}{a(t)}\right)^{\cdot}-\frac{p(t)}{a(t)\,\psi(x(t))}\right]\omega_1(t)\\&\quad-\frac{1}{\rho(t)}\frac{f'(x(t))}{\psi(x(t))}\omega_1^2(t)+\rho^{\cdot\cdot}(t)\,F(x(t)),\end{aligned}$$

where $\omega_1(t)=(\rho(t)\,\psi(x(t))\,x^{\cdot}(t))/f(x(t))$ so we have

$$\begin{aligned}\omega^{\cdot\cdot}(t)=&-\frac{\rho(t)}{a(t)}q(t)+\frac{\left[\dfrac{\rho^{\cdot}(t)}{\rho(t)}+\dfrac{a(t)}{\rho(t)}\left(\dfrac{\rho(t)}{a(t)}\right)^{\cdot}-\dfrac{p(t)}{a(t)\,\psi(x(t))}\right]^{\cdot 2}}{4\left(\dfrac{1}{\rho(t)}\dfrac{f'(x(t))}{\psi(x(t))}\right)}\\&-\left[\left(\frac{1}{\rho(t)}\frac{f'(x(t))}{\psi(x(t))}\right)^{1/2}\omega_1(t)-\frac{\left[\dfrac{\rho^{\cdot}(t)}{\rho(t)}+\dfrac{a(t)}{\rho(t)}\left(\dfrac{\rho(t)}{a(t)}\right)^{\cdot}-\dfrac{p(t)}{a(t)\,\psi(x(t))}\right]}{2\left(\dfrac{1}{\rho(t)}\dfrac{f'(x(t))}{\psi(x(t))}\right)^{1/2}}\right]^2\\&+\rho^{\cdot\cdot}(t)\,F(x(t)).\end{aligned}$$

Using (8), we have

$$\begin{aligned}\omega^{\cdot\cdot}(t)\leqslant&-\frac{\rho(t)}{a(t)}q(t)+\frac{\rho(t)}{4}\frac{\psi(x(t))}{f'(x(t))}\\&\times\left[\frac{\rho^{\cdot}(t)}{\rho(t)}+\frac{a(t)}{\rho(t)}\left(\frac{\rho(t)}{a(t)}\right)^{\cdot}-\frac{p(t)}{ca(t)}\right]^{\cdot}+\rho^{\cdot\cdot}(t)\,F(x(t))\\=&-\frac{\rho(t)}{a(t)}q(t)+\frac{\rho^{\cdot\cdot}(t)\,\psi(x(t))}{f'(x(t))}\\&\times\left[\frac{F(x(t))\,f'(x(t))}{\psi(x(t))}+\frac{\rho(t)}{4\rho^{\cdot\cdot}(t)}\left(\frac{\rho^{\cdot}(t)}{\rho(t)}+\frac{a(t)}{\rho(t)}\left(\frac{\rho(t)}{a(t)}\right)^{\cdot}-\frac{p(t)}{ca(t)}\right)^2\right].\end{aligned}$$

By (19), we obtain

$$\omega^{\cdot\cdot}(t) \leqslant -\frac{\rho(t)}{a(t)} q(t). \tag{21}$$

Integrating (21) twice over $[t_1, t)$, $t_1 \geqslant t_0$ we have

$$\int_{t_1}^{t} \int_{t_1}^{s} \frac{\rho(\tau)}{a(\tau)} q(\tau)\, d\tau\, ds \leqslant \omega(t_1) + (t - t_1)\, \omega^{\cdot}(t_1). \tag{22}$$

After dividing (22) by t we take the upper limit as $t \to \infty$ to arrive at

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_1}^{t} \int_{t_1}^{s} \frac{\rho(\tau)}{a(\tau)} q(\tau)\, d\tau\, ds \leqslant \omega^{\cdot}(t_1) < \infty$$

which contradicts (20). Hence the proof is complete.

For illustration we consider

EXAMPLE 3. The equations

$$(t^{4/3} x^{\cdot})^{\cdot} \pm \frac{\cos t}{4t} x^{\cdot} + (t^2 \sin t)\, x^{1/3} = 0, \qquad t > 0, \tag{23}$$

$$(t^{4/3}(1 + x^2)\, x^{\cdot})^{\cdot} \pm \frac{\sin t}{4t} x^{\cdot} + (t^2 \sin t)\, x^{1/3} = 0, \qquad t > 0 \tag{24}$$

and

$$(t^{4/3}(2 - \sin x)\, x^{\cdot})^{\cdot} \pm \frac{\cos t}{4t} x^{\cdot} + (t^2 \cos t)\, x^{1/3} = 0, \qquad t > 0 \tag{25}$$

are oscillatory by Theorem 2 for $\rho(t) = t^{2/3}$, whereas we believe that none of the results in [1, 2, 6, 7, 8] can be applied to any one of these equations.

The following theorem is concerned with the asymptotic behavior of solutions of equation (2).

THEOREM 3. *Let* b: $[t_0, \infty) \to R$ *be continuous,* $\psi(x) \geqslant c > 0$ *and* $f'(x) > 0$ *for* $x \neq 0$ *and let condition* (17) *hold. If there is a twice differentiable function* ρ: $[t_0, \infty) \to (0, \infty)$ *such that conditions* (18) *and* (19) *hold and, in addition,*

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \frac{\rho(\tau)}{a(\tau)} [q(\tau) - c_1 |b(\tau)|]\, d\tau\, ds = \infty, \tag{26}$$

for every $c_1 > 0$, *then for all solutions* $x(t)$ *of Eq.* (2), *we have*

$$\lim_{t \to \infty} \inf x(t) = 0.$$

Proof. Let $x(t)$ be a solution of (2) defined on an interval $[t_1, \infty)$, $t_1 \geqslant t_0$, with $\lim_{t \to \infty} \inf |x(t)| > 0$. We observe that $x(t)$ is always non-oscillatory and hence we suppose without loss of generality that $x(t) \neq 0$ for all $t \geqslant t_1$. Next, we define

$$\omega(t) = \rho(t)\, F(x(t)).$$

As in the proof of Theorem 2, we have

$$\omega^{\cdot\cdot}(t) \leqslant -\frac{\rho(t)}{a(t)}\, q(t) + \frac{\rho(t)}{a(t)} \frac{b(t)}{f(x(t))} + \frac{\rho(t)}{4} \frac{\psi(x(t))}{f'(x(t))}$$

$$\times \left[\frac{\rho^{\cdot}(t)}{\rho(t)} + \frac{a(t)}{\rho(t)} \left(\frac{\rho(t)}{a(t)} \right)^{\cdot} - \frac{p(t)}{ca(t)} \right]^2 + \rho^{\cdot\cdot}(t)\, F(x(t))$$

$$= -\frac{\rho(t)}{a(t)}\, q(t) + \frac{\rho(t)}{a(t)} \frac{b(t)}{f(x(t))} + \frac{\rho^{\cdot\cdot}(t)\, \psi(x(t))}{f'(x(t))}$$

$$\times \left[\frac{F(x(t))\, f'(x(t))}{\psi(x(t))} + \frac{\rho(t)}{4\rho^{\cdot\cdot}(t)} \left[\frac{\rho^{\cdot}(t)}{\rho(t)} + \frac{a(t)}{\rho(t)} \left(\frac{\rho(t)}{a(t)} \right)^{\cdot} - \frac{p(t)}{ca(t)} \right]^2 \right].$$

Using (19), we get

$$\omega^{\cdot\cdot}(t) \leqslant -\frac{\rho(t)}{a(t)}\, q(t) + \frac{\rho(t)}{a(t)} \frac{|b(t)|}{|fc_1)|}, \tag{27}$$

where $c_1 = \inf_{t \geqslant t_1} x(t)$ for $x > 0$, $c_1 = \sup_{t \geqslant t_1} x(t)$ for $x < 0$. Integrating (27) twice over $[t_1, t)$, $t_1 \geqslant t_0$ we have

$$\int_{t_1}^{t} \int_{t_1}^{s} \frac{\rho(\tau)}{a(\tau)} \left[q(\tau) - \frac{1}{|f(c_1)|}\, |b(\tau)| \right] d\tau\, ds \leqslant \omega(t_1) + (t - t_1)\, \omega^{\cdot}(t_1). \tag{28}$$

If we divide (28) by t and take limit superior as $t \to \infty$, we obtain the desired contradiction. This completes the proof.

Remark. Our Theorem 3 is stronger and more general than Theorem 4 in [8]. To see this consider the equation

$$x^{\cdot\cdot}(t) + t^{-1} x^{1/3}(t) = 2t^{-3} + t^{-4/3}, \qquad t > 0. \tag{29}$$

It is easy to check that the hypotheses of Theorem 3 are satisfied for $\rho(t)=t^{1/3}$ and hence every solution x of Eq. (15) satisfies $\lim_{t\to\infty}\inf x(t)=0$. One such solution is $x(t)=1/t$. We note that Theorem 4 in [8] fails to apply to Eq. (15), since

$$\int^{\infty} s^{1/3}[2s^{-3}+\dot{s}^{-4/3}]\,ds=\infty.$$

The following corollary is an immediate consequence of Theorem 3, and it includes Theorem 4 in [8] as a special case.

COROLLARY 1. *Let condition* (26) *in Theorem* 3 *be replaced by*

$$\lim_{t\to\infty}\sup\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\frac{\rho(\tau)}{a(\tau)}\,q(\tau)\,d\tau\,ds=\infty \tag{30}$$

and

$$\int^{\infty}\frac{\rho(s)}{a(s)}\,|b(s)|\,ds<\infty, \tag{31}$$

then the conclusion of Theorem 3 *holds.*

The following examples are illustrative.

EXAMPLE 4. The equation

$$\begin{aligned}(t(1+x^2)\,x^{\cdot})^{\cdot}+t^{-1/3}x^{1/3}&=t^{-1/3}\sin^{1/3}(\ln t)-\frac{1}{t}\sin(\ln t)\\&\quad\times[1-\sin^2(\ln t)+2\cos^2(\ln t)],\end{aligned} \tag{32}$$

for $t>0$, has an oscillatory solution $x(t)=\sin(\ln t)$, $\lim_{t\to\infty}\inf \sin(\ln t)\neq 0$. All conditions of Corollary 1 are satisfied for $\rho(t)=(t)^{1/3}$ except condition (31).

EXAMPLE 5. Consider the equation

$$\begin{aligned}&(t(1+x^2)\,x^{\cdot})^{\cdot}-(t^{1/3}+t^{1/9})\,x^{\cdot}+\tfrac{1}{3}\,t^{-1/3}x^{1/3}\\&\quad=\tfrac{1}{9}\,t^{-2/3},\qquad t>0.\end{aligned} \tag{33}$$

We let $\rho(t)=t^{1/3}$. Conditions (18) and (19) of Corollary 1 are violated. Equation (33) has the nonoscillatory solution $x(t)=t^{1/3}\to\infty$ as $t\to\infty$.

EXAMPLE 6. The equations

$$(t(1+|x|^{\alpha})\,x^{\cdot})^{\cdot} - \tfrac{1}{3}x^{\cdot} + t^{-2/3}|x|^{\alpha}\operatorname{sgn} x$$
$$= \frac{4}{3t^2} + \frac{1+\alpha}{t^{2+\alpha}} + t^{-(2/3)-\alpha}, \tag{34}$$

for $\frac{1}{2} \leqslant \alpha < 1$ and $t > 0$ and

$$(t(2-\sin x)\,x^{\cdot})^{\cdot} + \tfrac{1}{3}x^{\cdot} + t^{-(/3)}x^{1/3}$$
$$= \frac{1}{t^2}\left[t + \frac{5}{3} - \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right], \qquad t > 0 \tag{35}$$

have a nonoscilllatory solution $x(t) = 1/t \to 0$ as $t \to \infty$.

All conditions of Theorem 3 are satisfied.

Remark. The results of this paper can be extended, without any change in the hypotheses, to equations of the form

$$(a(t)\,\psi(x)\,x^{\cdot})^{\cdot} + p(t)\,x^{\cdot} + F(t, x) = 0, \tag{36}$$

where F: $[t_0, \infty)\, xR \to R$ is continuous. Here we assume that there is a differentiable function f: $R \to R$ such that

$$\frac{F(t, x)}{f(x)} \geqslant q(t) \qquad \text{for} \quad t \geqslant t_0 \quad \text{and} \quad x \neq 0.$$

The functions a, p, q, ψ, and f are as given above. Accordingly, the equations

$$(t^{4/3}(1+x^2)\,x^{\cdot})^{\cdot} \pm \frac{\sin t}{4t}x^{\cdot} + t^{-1/3}x^{1/3}\exp(\sin x) = 0, \quad t > 0 \tag{37}$$

and

$$(t^{4/3}(2-\sin x)\,x^{\cdot})^{\cdot} \pm \frac{\cos t}{t}x^{\cdot} + t^{-1/3}x^{1/3}\ln(e+x^2) = 0, \quad t > 0 \tag{38}$$

are oscillatory by Theorem 2 for $\rho(t) = t^{2/3}$.

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